THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2060B Mathematical Analysis II (Spring 2017)
HW3 Solution

Yan Lung Li

1. (P.179 Q3) We proceed by induction on n:

   Base step $n = 1$: This reduces to usual Leibniz rule (6.13(c))

   Inductive step: Suppose for some $N \in \mathbb{N}$, the statement holds for all $n < N$. When $n = N$ (the variable x is suppressed for simplicity)

$$(fg)^{(N)} = ((fg)')^{(N-1)} = (f'g + fg')^{(N-1)} = (f'g)^{(N-1)} + (fg')^{(N-1)}$$

   now by inductive hypothesis for $n = N - 1$ on $f'g$ and fg' respectively, we have

$$\begin{aligned}
(f'g)^{(N-1)} + (fg')^{(N-1)} &= \sum_{k=0}^{N-1} \binom{N-1}{k} (f')^{(N-1-k)} g^{(k)} + \sum_{k=0}^{N-1} \binom{N-1}{k} f^{(N-1-k)} (g')^{(k)} \\
&= (f^{(N)} g + \sum_{k=1}^{N-1} \binom{N-1}{k} f^{(N-k)} g^{(k)}) + (\sum_{k=1}^{N-1} \binom{N-1}{k-1} f^{(N-k)} (g)^{(k)} + fg^{(N)}) \\
&= f^{(N)} g + \sum_{k=1}^{N-1} (\binom{N-1}{k} + \binom{N-1}{k-1}) f^{(N-k)} g^{(k)} + fg^{(N)} \\
&= f^{(N)} g + \sum_{k=1}^{N-1} \binom{N}{k} f^{(N-k)} g^{(k)} + fg^{(N)} \\
&= \sum_{k=0}^{N} \binom{N}{k} f^{(N-k)} g^{(k)}
\end{aligned}$$

   Therefore, the statement holds for $n = N$. Hence by induction the statement holds for all $n \in \mathbb{N}$.

2. (P.179 Q4) Consider $f(x) = \sqrt{1+x}$ for $x \geq 0$. Then f is twice differentiable with

$$f'(x) = \frac{1}{2\sqrt{1+x}}; f''(x) = -\frac{1}{4(1+x)^{\frac{3}{2}}}$$

   and hence for all $y > 0$, $0 > f''(y) > -\frac{1}{4}$

   Now given any $x > 0$, let $I = [0, x]$ and consider f defined on $[0, x]$; f, f' are continuous on $[0, x]$ and f'' exists on $(0, x)$. Apply Taylor's theorem (Theorem 6.4.1) with $x_0 = 0$, there exists $c \in (0, x)$ such that

$$f(x) = f(0) + f'(0)x + \frac{f''(c)}{2!} x^2$$

More explicitly, this implies

$$\sqrt{1+x} = 1 + \frac{1}{2}x + \frac{f''(c)}{2}x^2$$

Since $c > 0$ and $x^2 > 0$, $0 > \frac{f''(c)}{2}x^2 > -\frac{1}{8}x^2$, and therefore

$$1 + \frac{1}{2}x - \frac{1}{8}x^2 < \sqrt{1+x} < 1 + \frac{1}{2}x$$

3. (P.179 Q10) Let $h(x) = \begin{cases} e^{-\frac{1}{x^2}} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$, note that h is differentiable on $\mathbb{R}\backslash\{0\}$ with $h'(x) = \dfrac{2}{x^3}h(x)$. We proceed by establishing the following claims:
(i) for all $k \in \mathbb{N}$, $\lim\limits_{x\to 0} \dfrac{h(x)}{x^k} = 0$.

(ii) for all $n \in \mathbb{N}$, for all $k \in \mathbb{N}$, $\lim\limits_{x\to 0} \dfrac{h^{(n)}(x)}{x^k} = 0$.

(iii) for all $n \in \mathbb{N}$, n th derivative of h at 0 exists and $h^{(n)}(0) = 0$.

Proof of (i): Induction on k:

Base step $k = 1$: $\lim\limits_{x\to 0} \dfrac{h(x)}{x} = \lim\limits_{x\to 0} \dfrac{\frac{1}{x}}{e^{\frac{1}{x^2}}} = \lim\limits_{x\to 0} \dfrac{-\frac{1}{x^2}}{e^{\frac{1}{x^2}} \cdot -\frac{2}{x^3}} = \lim\limits_{x\to 0} \dfrac{x}{2e^{\frac{1}{x^2}}} = 0$
where we have applied L'Hospital's rule in the second equality (careful justifications are left as exercises for readers)

Inductive step: Suppose for some $K \in \mathbb{N}$, the statement holds for all $k < K$.

When $k = K$, $\lim\limits_{x\to 0} \dfrac{h(x)}{x^K} = \lim\limits_{x\to 0} \dfrac{\frac{1}{x^K}}{e^{\frac{1}{x^2}}} = \lim\limits_{x\to 0} \dfrac{-\frac{K}{x^{K+1}}}{e^{\frac{1}{x^2}} \cdot -\frac{2}{x^3}} = \dfrac{K}{2} \lim\limits_{x\to 0} \dfrac{h(x)}{x^{K-2}} = 0$
Again, we have applied L'Hospital's rule in the second equality.

Therefore, the statement holds for $k = K$. Hence by induction the statement holds for all $k \in \mathbb{N}$.

Proof of (ii): Induction on n:

Base step $n = 1$: for all $k \in \mathbb{N}$ $\lim\limits_{x\to 0} \dfrac{h'(x)}{x^k} = \lim\limits_{x\to 0} \dfrac{2h(x)}{x^{3+k}} = 0$ (by (i))

Inductive step: Suppose for some $N \in \mathbb{N}$, the statement holds for all $n < N + 1$.
When $n = N + 1$, $h^{(N+1)}(x) = (h'(x))^{(N)} = (\frac{2}{x^3}h(x))^{(N)}$
By generalised Leibniz rule (Section 6.4 Q3), we have

$$(\frac{2}{x^3}h(x))^{(N)} = \sum_{l=0}^{N} \binom{N}{l} (\frac{2}{x^3})^{(N-l)} h(x)^{(l)}$$

for each l, $(\frac{2}{x^3})^{(N-l)} = \dfrac{n_l}{x^{3+(N-l)}}$ for some $n_l \in \mathbb{Z}$, and hence we have

$$\sum_{l=0}^{N} \binom{N}{l} (\frac{2}{x^3})^{(N-l)} h(x)^{(l)} = \sum_{l=0}^{N} \binom{N}{l} \frac{n_l h(x)^{(l)}}{x^{3+N-l}}$$

Therefore, for all $k \in \mathbb{N}$, $\lim\limits_{x\to 0} \dfrac{h^{(N+1)}(x)}{x^k} = \lim\limits_{x\to 0} \dfrac{\sum_{l=0}^{N} \binom{N}{l} \frac{n_l h(x)^{(l)}}{x^{3+N-l}}}{x^k} = \lim\limits_{x\to 0} \sum_{l=0}^{N} \binom{N}{l} \dfrac{n_l h(x)^{(l)}}{x^{3+N-l+k}} = 0$ by inductive hypothesis.

Therefore, the statement holds for $n = N + 1$. Hence by induction the statement holds for all $n \in \mathbb{N}$.

Proof of (iii): Induction on n:

Base step $n = 1$: $\lim_{x \to 0} \dfrac{h(x) - h(0)}{x - 0} = 0$ by (i). Hence $h^{'}(0) = 0$.

Inductive step: Suppose for some $N \in \mathbb{N}$, the statement holds for all $n < N + 1$.
When $n = N + 1$, $\lim_{x \to 0} \dfrac{h^{(N)}(x) - h^{(N)}(0)}{x - 0} = \lim_{x \to 0} \dfrac{h^{(N)}(x)}{x} = 0$ by (ii) with $k = 1$. Therefore, $(N + 1)$ th derivative of h at 0 exists and $h^{(N+1)}(0) = 0$.

Therefore, the statement holds for $n = N + 1$. Hence by induction the statement holds for all $n \in \mathbb{N}$.

Now fix $x \neq 0$, $x_0 = 0$ and apply taylor's theorem (Theorem 6.4.1) to h, then for each $n \in \mathbb{N}$, $h(x) = P_n(x) + R_n(x)$.

By (iii), $h^{(l)}(0) = 0$ for all $l \in \mathbb{N}$. Therefore, $P_n(x) \equiv 0$, and hence $h(x) = R_n(x)$ for all $n \in \mathbb{N}$.

Since $h(x) \neq 0$, $R_n(x)$ does not converge to 0 as $n \to \infty$.

Remark. The key point of this question is to express $h^{(N+1)}$ in terms of a sum of its lower derivatives with rational functions as coefficients. Many students recognised this, but were not able to formulate this in precise term or providing enough justification for this.